The Maximum Degree of a Random Delaunay Triangulation in a Smooth Convex

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Abstract. We give a new polylogarithmic bound on the maximum degree of a random Delaunay triangulation in a smooth convex, that holds with probability one as the number of points goes to infinity. In particular, our new bound holds even for points arbitrarily close to the boundary of the domain.

Keywords: Maximum degree, Random, Delaunay triangulation

1 Introduction

The Delaunay triangulation is a fundamental data structure in computational geometry, with applications in shape reconstruction, interpolation, routing in networks and countless others. Many algorithms applied to the Delaunay triangulation are in some way dependent on the degree of the vertices, and so bounding the maximum degree of a triangulation implies useful bounds on the complexity of other geometric algorithms. For $\mathbf{X}$ a set of n points in $\mathbb{R}^2$, the maximum degree of the Delaunay triangulation of $\mathbf{X}$, $\mathrm{DT}(\mathbf{X})$ is easily seen to be $n-1$. For example, $n-1$ points may be arranged in a circle around a central *hub*, as in Figure 1. For most applications, this bound is far too pessimistic, and so it is desirable to seek bounds which are more realistic. One such option is to consider the maximum degree when the input points are provided uniformly at random. Let $\Delta_{\mathbf{X}}(x)$ denote the degree of $x \in \mathbf{X}$ in $\mathrm{DT}(\mathbf{X})$.

For $\Gamma a homogeneous Poisson process on the entire Euclidean plane with intensity 1, Bern et al. [2] give a proof to bound the expected maximum degree of any vertex of the Delaunay triangulation $\mathrm{DT}(\mathbf{\Gamma})$ falling within the box $[0, \sqrt{n}\,]^2$,

$$\mathbb{E}\left[\max_{x\in[0,\sqrt{n}\,]^2} \Delta_{\mathbf{\Gamma}}(x)\right] = \Theta\left(\frac{\log n}{\log\log n}\right). \tag{1}$$

This bound has a few properties that limit its utility in practice:

1. The bound implicitly avoids dealing with 'boundary effects'.

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2. The bound only gives the expectation.
3. It is only given for a set of points generated by a Poisson distribution.

To see the problems that arise near the boundary, we observe that the distribution of vertex degrees for a Delaunay triangulation near the convex hull is significantly skewed (see Figure 2), with most of the vertices in this area having high degree. Also affected are the lengths of edges close to the border, and we note in particular that edges within this region are asymptotically longer than those away from the border [1]. It is therefore not altogether trivial that the maximum degree should be bounded polylogarithmically in this case. We show that in fact it is.

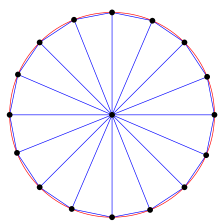

Fig. 1: Max degree worst case.

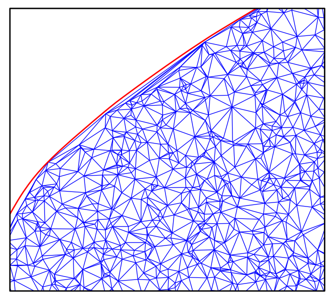

Fig. 2: Skewed degree distribution near boundary.

2 Contributions

We provide a new bound for the degree for the Delaunay triangulation. In particular, taking the set $\mathbf{X} := \{X_1, X_2, \cdots, X_n\}$ to be n uniformly distributed points in a smooth compact convex $\mathcal{D} \subset \mathbb{R}^2$, we demonstrate that for any $\xi > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{x\in\mathbf{X}} \Delta_{\mathbf{X}}(x) \geq \log^{2+\xi} n \right) = 0. \tag{2}$$

As an example application of our bound, we consider the *vertex deletion* operation, which removes a vertex, x, from $\mathrm{DT}(\mathbf{X})$ and outputs $\mathrm{DT}(\mathbf{X} \setminus x)$. It is well-known that this can be computed in $O(\Delta_{\mathbf{X}}(x))$ [4]. Unfortunately, the bound in Equation 1 is not sufficient to bound the worst-case run-time of this algorithm on a finite set of random points, since no information is given about points near the boundary. Our results guarantee that with probability tending to one, no vertex exists in the domain that can't be removed in fewer than $O(\log^{2+\xi} n)$ steps.

The proof is available in [3, Appendix].

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